Tarski Theorems on Self-Dual Equational Bases for Groups

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Abstract

We present independent self-dual equational bases of arbitrarily large finite sizes for the equational theory of groups treated as varieties of various well-known types. Here the dual of a term f is the mirror reflection of f. For each type of group theory, we provide an independent self-dual basis with n identities for $n = 2, 3, 4$. Then we develop a simple algorithmic procedure to construct independent self-dual equational bases of arbitrary finite sizes in such a way that the new larger equational bases depend explicitly on the initial bases

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of small sizes. Applying this “expansion” procedure, we show that every finitely based variety of groups can be defined by an independent self-dual set of n identities for all $n \geq 2$. Apart from generalizing the various theorems of Alfred Tarski who initiated this topic in the late 1960s, these proofs also provide explicitly the bases and hence may be construed as the first constructive proof of Tarski’s theorems as well.

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> *The view has become more and more common that the deductive method is the only essential feature by means of which the mathematical disciplines can be distinguished from all other sciences; not only is every mathematical discipline a deductive theory, but also, conversely, every deductive theory is a mathematical discipline.*
>
> Alfred Tarski

1 Introduction

Let K be a finitely based equational theory. Following Alfred Tarski [7, 8], we denote by $\nabla(K)$ the set of cardinalities of irredundant equational bases of K. More precisely,

$$\nabla(K) = \{n \mid K \text{ has an independent basis with } n \text{ identities}\}.$$

Tarski has shown that if K satisfies an equation of the form $f = x$ where f has at least two occurrences of variables, then $\nabla(K)$ is an unbounded interval of positive integers. This we call *Tarski’s unbounded theorem.* For such varieties, Tarski has further shown that if m and n belong to the set $\nabla(K)$, then every integer between m and n also belongs to $\nabla(K)$. This we call *Tarski’s interpolation theorem.*

Analogs of Tarski’s theorems do not necessarily hold, however, when syntactic constraints are placed on the the equations of the basis. For example, D. Kelly and R. Padmanabhan [2] showed a variety of algebras definable by an independent syntactically constrained basis with 2 and 4 identities, but having no such basis with 3 identities. The constraint for the example is that the basis is self-dual with respect to meet and join. In this paper we consider a natural self-duality constraint for group theory (GT) equations and prove analogs of Tarski's theorems.

Let us view GT, the equational theory of groups, as of type $\langle 2, 1\rangle$ (or of type $\langle 2, 1, 0\rangle$) with binary multiplication, say $\cdot$, unary inverse, say $'$, and possibly nullary e. The *dual* of the term $x \cdot y$, written $\widetilde{x \cdot y}$, is, by definition, the term $\widetilde{y} \cdot \widetilde{x}$; for inverse, $\widetilde{x'}$ is $\widetilde{x}\,'$; the duals of e and variables are themselves. Thus, for example, the dual of the composite term $x \cdot (z' \cdot (e \cdot t))$ is $((t \cdot e) \cdot z') \cdot x$. In other words, $\widetilde{f}$ is just the mirror reflection of the term f. Since the transpose of a group is also a group, the class of all groups is self-dual in this model-theoretic sense. So if an identity $f = g$ is valid in all groups, then its dual identity $\widetilde{f} = \widetilde{g}$ must also be valid in all groups. Let us call a set of identities Σ self-dual if $f = g \in \Sigma \Rightarrow \widetilde{f} = \widetilde{g} \in \Sigma$ (modulo, perhaps, renaming of variables and flipping the equality). In particular, an identity σ is self-dual if σ is the same as $\widetilde{\sigma}$. For example, the associative law $(xy)z = x(yz)$ is self-dual because its mirror image is $(zy)x = z(yx)$, which is, of course, the associative law itself.

In spite of an abundance of different axiomatic approaches to group theory, no independent equational basis that is also self-dual was previously known to exist. In addition to providing one such basis for every finitely based variety of GT, we prove the self-dual analogs of Tarski's unbounded and interpolation theorems for group theory. In particular, we show that $\nabla_{sd}(K) = [2, \omega)$ for all group theories K under various treatments, where now $\nabla_{sd}(K)$ denotes the set of all cardinalities of irredundant and self-dual equational bases of K. Additionally, we mention some open problems.

Tarski's original 1975 proof is topological and hence existential in nature. In sharp contrast, the proof presented here is completely constructive, and the equational bases of larger sizes depend explicitly on the initial equational axioms. Thus, these irredundant equational bases may be construed as the first constructive proofs of Tarski's original theorems.

Related work can be found in [6], where it was shown that Boolean algebras, distributive lattices, and modular lattices all have independent self-dual bases. In [2] it is shown that every finitely based self-dual variety of lattices has an independent self-dual basis with n identities for all $n \geq 4$.

In [5] we presented several of the 2-, 3-, and 4-bases on which this work is based.

2 Self-Dual Equational Bases for GT

We begin by providing an independent self-dual 2-basis for GT.

Theorem 1. *The pair of equations*

$$\begin{aligned} ((xy)z)(yz)' &= x \\ (xy)'(x(yz)) &= z \end{aligned} \tag{A}$$

constitute a minimal self-dual basis for group theory.

Proof. Let $xy = uy$. Then $x = ((xy)z)(yz)' = ((uy)z)(yz)' = u$. Thus, by duality, we have both cancellation laws. Now substitute $x = t(yz)'$ in the first member of (A). Cancelling the common term $(yz)'$ on the right, we obtain $((t(yz)')y)z = t$ (1), and hence the algebra is a quasigroup, that is; given a and b, the equation $xa = b$ is soluble uniquely and dually (i.e., $ay = b$ is soluble uniquely). Plugging $t = yz$ in (1) and cancelling z, we get $((yz)(yz)')y = y$. Since we have a quasigroup, given y, we choose z such that $yz = t$. This choice is possible by the dual of (1). Notice that the variable y remains untouched in this process. Thus we have $(tt')y = y$. So the element (tt') is independent of t (i.e., $tt' = uu' = e$, say). Hence we have both an identity element and the inverse property. Moreover, by cancellation, $t'' = t$. All we have to do now is to derive associativity. We substitute $z = e$ in the original equation of (A) to obtain $(xy)y' = x$. Finally, postmultiplying the first equation of (A) by $(yz)'' = (yz)$, we get the desired associative law $(xy)z = x(yz)$. See the appendix for an automated proof obtained by the theorem-proving program OTTER [3]. Independence is clear because $xy = x$ (left projection) models the first axiom in (A) but not the second. □

Corollary 1. *A minimal self-dual basis for the theory of Abelian groups is*

$$\begin{aligned} (x(yz))(zx)' &= y \\ (xy)'((yz)x) &= z. \end{aligned} \tag{AG}$$

Proof. By Theorem 1, it is enough if we derive the commutativity from (AG). This derivation is straightforward. For completeness, we have included in the appendix a formal proof obtained by OTTER for commutativity. The rest follows from Theorem 1.

Independence: Define $xy = ix - y$ and $x' = x$ in complex numbers. Then the first axiom in (AG) is valid, but "$\cdot$" is not commutative, and hence the first one does not imply the second axiom. By duality, (AG) is an independent basis for the theory of all Abelian groups. $\square$

See Tables 3 and 2 of Section 5 for 3-bases and 4-bases for GT and AG.

2.1 Automated Search for an Independent Self-Dual 2-Basis for GT

The sought-after system was in terms of product and inverse. Every such system for group theory must have an equation with at least three variables, an equation with a just variable on one side, and an equation with inverse, and each variable must have an even number of occurrences. Hence the smallest candidates have the form $t(x, y, z) = z$, where $t(x, y, z)$ has one occurrence of inverse, two occurrences each of x and y, and one occurrence of z. Exactly 30 group identities (15 pairs of duals) have these properties (five associations each of $x(yx)'yz = z$, $(xy)'xyz = z$, and $xy(xy)'z = z$ and their duals). An OTTER search was run on each of the 15 candidate systems, and three were found axiomatize group theory. Each basis is independent. Systems for Abelian groups must also satisfy the above syntactic properties, and the same lower bound on the size applies. In this case, there are 120 Abelian group identities (60 candidate pairs). (All associations (5) of all permutations (24) of $\{x, y, (xy)', z\}$ set equal to z, which includes duals.) Rather than run the analogous OTTER searches on the 60 candidate pairs, we simply took system (A) for ordinary groups and ran several OTTER searches with various corresponding terms commuted. On the second search, the self-dual system for Abelian groups was found.

2.2 Groups as Birkhoff Quasigroups (BQ)

We conclude this section by giving a quasigroup approach to GT [1, p. 163]. This is based on the well-known fact that associative quasigroups are just groups. Consider algebras of type $\langle 2, 2, 2\rangle$ with three binary operations $x \cdot y$, x/y, $x\backslash y$, where

$x \cdot y$	is group multiplication
$x/y = x \cdot y'$	is right division
$x\backslash y = x' \cdot y$	is left division.

The duality map is given by $x/y \leftrightarrow y\backslash x$ and $x \cdot y \leftrightarrow y \cdot x$.

We exploit the self-dual identity $x/(y\backslash z) = (z/x)\backslash y$ connecting the two divisions. This identity is due to Garrett Birkhoff. For this reason, we call this the Birkhoff identity and the variety as Birkhoff quasigroups (BQ).

Theorem 2. *A quasigroup $\langle G; /, \cdot, \backslash\rangle$ satisfying the self-dual Birkhoff identity $x/(y\backslash z) = (z/x)\backslash y$ is a group with $\cdot$ as group multiplication, $x/y = x \cdot y'$ and $x\backslash y = x' \cdot y$.*

Proof. Consider the following Otter derivation.

2	$y(y\backslash x) = x$	[axiom]
3	$(x/y)y = x$	[axiom]
4	$y\backslash(yx) = x$	[axiom]
5	$(xy)/y = x$	[axiom]
6	$x/(y\backslash z) = (z/x)\backslash y$	[axiom]
11,10	$x/(y\backslash x) = y$	[2 (1) $\rightarrow$ 5 (1.1)]
13	$x/y = ((zy)/x)\backslash z$	[4 (1) $\rightarrow$ 6 (1.2)]
15,14	$(x/x)\backslash y = y$	[10 (1) $\rightarrow$ 6 (1)]
16	$(x/(y(z\backslash x)))\backslash z = y$	[5 (1) $\rightarrow$ 6 (1)]
22	$((x/y)\backslash z)(z\backslash x) = y$	[6 (1) $\rightarrow$ 3 (1.1)]
24	$(x\backslash x)\backslash y = y$	[6 (1) $\rightarrow$ 14 (1.1),11]
34	$x/x = y\backslash y$	[24 (1) $\rightarrow$ 10 (1.2)]
38	$x\backslash x = y\backslash y$	[34 (1) $\rightarrow$ 34 (1)]
45	$x(y\backslash y) = x$	[38 (1) $\rightarrow$ 2 (1.2)]
71	$(((xy)/z)\backslash x)y = z$	[13 (1) $\rightarrow$ 3 (1.1)]
75	$(x/(yx))\backslash(z/z) = y$	[14 (1) $\rightarrow$ 16 (1.1.2.2)]
116	$(x\backslash y)(y\backslash z) = x\backslash z$	[10 (1) $\rightarrow$ 22 (1.1.1)]
331	$xy = (z/(xz))\backslash y$	[75 (1) $\rightarrow$ 116 (1.1),15]
332	$(x/(yx))\backslash z = yz$	[331]
418	$(xy)z = x(yz)$	[332 (1) $\rightarrow$ 71 (1.1)]

Lines 2–5 are the quasigroup properties, and line 6 is the Birkhoff identity. Since associativity is now a consequence (as seen in line 418), we have the full group theory. Indeed, line 34 shows that $x/x = y\backslash y$ (= e, say). From line 45, we have $xe = x$. Finally, from line 2 we get $x(x\backslash e) = e$ and hence $x\backslash e = x'$. This completes the proof that $x/y = xy'$, the right division, and $x\backslash y = x'y$, the left division. $\square$

See Table 4 of Section 5 for irredundant self-dual 2-bases, 3-bases, and 4-bases for BQ.

3 Tarski Theorems for Self-Dual Group Varieties

In this section we develop a simple algorithmic procedure to construct independent self-dual equational bases of arbitrary finite sizes for group theory in such a way that the new, larger equational bases depend explicitly on the initial bases of small sizes. The method is general enougn to apply to any "nice" equational theory admitting absorption laws.

Theorem 3. *Let K be the equational theory of algebras of type $\langle 2, 1\rangle$ defined by the self-dual pair of identities*

$$\begin{array}{c} x * (x' * y) = y \\ (y * x') * x = y. \end{array}$$

Then $2n \in \nabla_{sd}(K)$ for all n, where $\nabla_{sd}(K)$ is the self-dual spectrum of K, that is,

$$\nabla_{sd}(K) = \{n \mid K \text{ has an independent self-dual basis with } n \text{ identities}\}.$$

Let us first prove the following lemma demonstrating why 4 belongs to $\nabla_{sd}(K)$. From this the reader can easily construct a basis with $2n$ identities for all n.

Lemma 1. $4 \in \nabla_{sd}(K)$.

Proof. Consider the following self-dual set of four identities:

$$x * (x' * (x * (x' * y)))) = y \tag{A}$$

$$x * (x' * (x * (x' * (x * (x' * y))))) = y \tag{B}$$

$$((((y * x') * x) * x') * x = y \tag{$\widetilde{\text{A}}$}$$

$$(((((y * x') * x) * x') * x) * x') * x = y. \tag{$\widetilde{\text{B}}$}$$

Indeed,

$$\begin{array}{lll} x * (x' * y) & = x * (x' * (x * (x' * (x * (x' * y))))) & \text{by } (A) \\ & = y & \text{by } (\widetilde{\text{A}}). \end{array}$$

Similarly, by the dual argument, we obtain $(y * x') * x = y$. Thus these four identities do form a basis for K. To show their independence, take an Abelian group of exponent 6, and interpret $x * y = x - y$ and $x' = x$.

Then $x * (x' * y) = x - (x - y) = y$, and hence both (A) and (B) are automatically valid here. Consider the meaning of $(\widetilde{\mathrm{B}})$ in this model:

$$(((((y * x') * x) * x') * x) * x') * x = y - 6x = y - 0 = y.$$

Thus $(\widetilde{\mathrm{B}})$ is also true in this model. But $(\widetilde{\mathrm{A}})$ will fail because $y - 4x \neq y$, since $4 \neq 0 \pmod 6$.

Similarly, taking an Abelian group of exponent 4 and interpreting $x*y = x - y$ and $x' = x$, we see (A), $(\widetilde{\mathrm{A}})$, and (B) are valid but $(\widetilde{\mathrm{B}})$ will fail. It is now clear that these four identities are independent. Thus $4 \in \nabla_{sd}(K)$. This completes the proof of Lemma 1. $\square$

Before going further, let us summarize what has happened: 4 and 6 are such that $gcd(4, 6) = 2$ and the crucial identity $x * (x' * y) = y$ has two occurrences of x and one occurrence of y on the left-hand side while the right-hand side has one y and no occurrences of x. Thus the identity $x * (x' * y) = y$ tells us, in an informal sense, that "2 = 0", and it is this property we exploit systematically to build independent bases of arbitrarily large sizes.

Now let us do this more formally. Define

$$\begin{aligned} \sigma(y) &= x * (x' * y) \\ \sigma^{k+1}(y) &= \sigma(\sigma^k(y)) = x * (x' * (\sigma^k(y)) \end{aligned}$$

and, dually, $\tau(y)$.

Lemma 2. *(The gcd lemma.) Let Σ be a set of identities of the form $\{\sigma^{ki}(y) = y, i = 1, 2, \cdots, m\}$. Define $S(\Sigma) = \{n|\sigma^n(y) = y \in \Sigma\}$. We claim that if $m, n \in S(\Sigma)$, then $gcd(m, n) \in S(\Sigma)$.*

Proof. Let $m, n \in S(\Sigma)$ with, say $m > n$. So $m = k + n$ for some positive integer k. Now

$$\begin{aligned} y &= \sigma^m(y) && \text{since } m \in S(\Sigma) \\ &= \sigma^{k+n}(y) && \text{since } m = k + n \\ &= \sigma^k(\sigma^n(y)) && \text{by definition of } \sigma \\ &= \sigma^k(y) && \text{since } n \in S(\Sigma). \end{aligned}$$

This computation shows that $m, n \in S(\Sigma)$ implies that $k = m - n \in S(\Sigma)$. So the set $S(\Sigma)$ is closed for all meaningful linear combinations of its members. In particular, the number set $S(\Sigma)$ is closed for gcd. The equational deduction made may be seen in this numerical setup as follows.

$$4, 6 \in S(\Sigma) \text{ and } gcd(4, 6) = 2; \text{ hence } 2 \in S(\Sigma) \text{ as well.}$$

$\square$

Now let us return to the proof of Theorem 3. Let K be the equational theory of all algebras of type $\langle 2, 1\rangle$ defined by the two identities

$$x * (x' * y) = y$$
$$(y * x') * x = y.$$

We wish to show that K can be defined by an independent self-dual set of n identities for all even numbers n, that is, $2n \in \nabla_{sd}(K)$ for all $n \geq 1$.

The two given identities are clearly duals of each other, and projection models demonstrate their independence. Thus $2 \in \nabla_{sd}(K)$.

To show that $2n \in \nabla_{sd}(K)$, simply choose the first n odd prime numbers $p_1, p_2, \cdots, p_n$. For $j = 1, 2, \cdots, n$ define the number q_j as Πp_i with $i = 1, 2, \cdots, j-1, j+1, \cdots, n$. It is now clear that $gcd(q_1, q_2, \cdots, q_n) = 1$ but no proper subset of $\{q_1, q_2, \cdots, q_n\}$ has this property.

For example, let us demonstrate that $6 \in \nabla_{sd}(K)$. Here $n = 3$ and $q_1 = 5 \times 7 = 35$, $q_2 = 3 \times 7 = 21$, and $q_3 = 3 \times 5 = 15$.

We clearly have $gcd(35, 21, 15) = 1$, but $gcd(35, 21) = 7$, $gcd(21, 15) = 3$, and $gcd(35, 15) = 5$. To show that $6 \in \nabla_{sd}(K)$, we choose the following self-dual set $\Sigma(6)$ of six identities:

$$\Sigma(6) = \{\sigma^i(y) = y,\ \tau^i(y) = y \mid i = 35, 21, 15\}.$$

Since $35, 21, 15 \in \Sigma(6)$, by the gcd lemma, we get that $gcd(35, 21, 15) = 1 \in \Sigma(6)$. This is just a fancy way of saying that we do have the two identities $x * (x' * y) = y$, $(y * x') * x = y$ as consequences of $\Sigma(6)$. In other words, $\Sigma(6)$ is indeed a self-dual basis for K. Independence is now obvious by the very construction of the three numbers q_1, q_2, q_3: to demonstrate the independence of say, $\tau^{15}(y) = y$ from the remaining five identities, we take the cyclic group $Z[14]$ and interpret $x * y = x - y \pmod{14}$, $x' = x$.

In this model $x * (x' * y) = x - (x - y) = y$, and hence all the three "σ" identities are automatically valid. By the definition of τ, we have

$\tau(y) = (y * x') * x = y - 2x$ and $\tau^i(y) = y - 2ix$. Thus $\tau^i(y) = y$ in the group $Z[14]$ if and only if 14 divides $2i$. Now 14 divides 70 and 42 but not 30. Hence in this group model, except for $\tau^{15}(y) = y$, all the other five identities of $\Sigma(6)$ are valid. This completes proof that $6 \in \nabla_{sd}(K)$.

The same argument applies to any even number: To show that $2n \in \nabla_{sd}(K)$, we simply take the appropriate group models based on the integers $q_1, q_2, \cdots, q_n$ as defined above and interpret $x * y = x - y$, $x' = x$ or $x * y = x + y$, $x' = -x$ as the case may be. We note that to prove the independence of the various σ-identities, we need the dual group model $x * y = x + y$, $x' = x$. The proof of Theorem 3 is now complete.

Theorem 4. *$2, 3, 4 \in \nabla_{sd}(GT)$.*

Proof. Here GT is a finitely based variety of groups. Let the type be $\langle 2, 1\rangle$ with group multiplication $x * y$ and group inverse $x \to x'$. Numbers 2, 3, and 4, are achieved by the following formal reduction schema:

$$\text{for } 2 : \left\{ \begin{array}{l} ((x * y) * (a * z)) * (y * (a * z))' = x \\ ((x * b) * y)' * ((x * b) * (y * z)) = z \end{array} \right\}$$

$$\text{for } 3 : \left\{ \begin{array}{l} x * (x' * y) = y \\ (x * y') * y = x \\ ((x * a) * y) * (b * z) = (x * a) * (y * (b * z)) \end{array} \right\}$$

$$\text{for } 4 : \left\{ \begin{array}{l} x * (x' * y) = y \\ (x * y') * y = x \\ x' * x = y * y' \\ ((x * (a * y)) * b) * z = x * (a * ((y * b) * z)) \end{array} \right\}.$$

OTTER proofs are given in the appendix. $\square$

Now we deal with higher odd numbers.

Theorem 5. *(The odd number case.) $2n + 1 \in \nabla_{sd}(GT)$ for all $n \geq 1$.*

Proof. Start from the 3-basis for K, namely,

$$x * (x' * y) = y$$
$$(x * y') * y = x$$
$$((x * a) * y) * (b * z) = (x * a) * (y * (b * z)).$$

Now simply blow up the first two self-dual identities $x * (x' * y) = y, (x * y') * y = x$ to an even number $2n$, as done in Theorem 3. By the very construction, this new self-dual basis consisting of $2n + 1$ identities does characterize GT and is obviously independent. The same models used in the corresponding constructions work here as well. $\square$

Theorem 6. *(The even number case.)* $2n \in \nabla_{sd}(GT)$ *for all* $n \geq 1$.

Proof. Use the four-basis

$$\begin{aligned}
&x * (x' * y) = y \\
&(x * y') * y = x \\
&x' * x = y * y' \\
&((x * (a * y)) * b) * z = x * (a * ((y * b) * z))
\end{aligned}$$

in the above argument and again just blow up the pair

$$\begin{aligned}
&x * (x' * y) = y \\
&(x * y') * y = x
\end{aligned}$$

to any desired even number, say, $2n - 2$. Hence the new set by adding the last two, namely,

$$\begin{aligned}
&x' * x = y * y' \\
&((x * (a * y)) * b) * z = x * (a * ((y * b) * z)),
\end{aligned}$$

gives an independent self-dual basis with $2n$ identities. $\square$

This completes the proof of Tarski unbounded theorem for groups: $\nabla_{sd}(GT) = [2, \omega)$.

4 Open Problems

As in the case of groups, the set of all identities true in a Moufang loop is closed for the duality of mirror reflection. The problems are to prove self-dual Tarski theorems for (1) the variety of all Moufang Loops and (2) the variety of all commutative Moufang loops.

Various varieties of complemented lattice can be viewed as algebras of type $\langle 2 \rangle$ with the Sheffer stroke operation. The duality here is mirror reflection. The problems are to prove self-dual Tarski theorems for the following varieties of type $\langle 2 \rangle$: (3) Boolean algebras, (4) orthomodular lattices, and (5) ortholattices.

5 2-Bases, 3-Bases, and 4-Bases

This section contains independent self-dual bases for several group varieties of various types. In each case, the 3-basis and the 4-basis can be used to prove theorems similar to Theorems 5 and 6, showing that $\nabla_{sd}(K) = [2, \omega)$.

Along with each basis is listed a justification of independence of one of the following types

1. Projection: there is a model satisfying $xy = x$ or $yx = x$.
2. Dual: the dual of the preceding justification.
3. 3-variable law: a 3-variable law is required to capture associativity.
4. $xy = -x - y$, e.g.: Abelian group or ring countermodel.
5. n-model: the program MACE [4] found an n-element countermodel.

Table 1: Boolean Groups

	Axioms		Independence
2-basis	$((xy)z)(yz) = x$	(BG2a)	projection
	$(zy)(z(yx)) = x$	(BG2b)	dual
3-basis	$y(yx) = x$	(BG3a)	projection
	$(xy)y = x$	(BG3b)	dual
	$(xy)z = x(yz)$	(BG3c)	3-variable law
4-basis	$y(yx) = x$	(BG4a)	$xy = y - x$
	$(xy)y = x$	(BG4b)	dual
	$xx = yy$	(BG4c)	$xy = -x - y$
	$(xy)(zu) = (xz)(yu)$	(BG4d)	3-variable law

Table 2: Abelian Groups

	Axioms		Independence
2-basis	$(y(xz))(zy)' = x$	(AG2a)	3-model
	$(yz)'((zx)y) = x$	(AG2b)	dual
3-basis	$y'(yx) = x$	(AG3a)	$xy = y - x, x' = x$
	$(xy)y' = x$	(AG3b)	dual
	$x((y(uu'))v) = (v((w'w)x))y$	(AG3c)	3-variable law
4-basis	$y'(yx) = x$	(AG4a)	$xy = y - x, x' = x$
	$(xy)y' = x$	(AG4b)	dual
	$xx' = y'y$	(AG4c)	$xy = -x - y, x' = x$
	$(xy)(zu) = (xz)(yu)$	(AG4d)	3-variable law

Table 3: Group Theory

	Axioms		Independence
2-basis	$((xy)z)(yz)' = x$	(GT2a)	projection
	$(zy)'(z(yx)) = x$	(GT2b)	dual
3-basis	$y(y'x) = x$	(GT3a)	projection
	$(xy')y = x$	(GT3b)	dual
	$(xy)z = x(yz)$	(GT3c)	3-variable law
4-basis	$y(y'x) = x$	(GT4a)	$xy = y - x, x' = x$
	$(xy')y = x$	(GT4b)	dual
	$x'x = yy'$	(GT4c)	$xy = -x - y, x' = x$
	$((x(uy))u)z = x(u((yu)z))$	(GT4d)	3-variable law

Table 4: Birkhoff Quasigroups

	Axioms		Independence
2-basis	$y(z((yz)\backslash x)) = x$	(BQ2a)	projection
	$((x/(zy))z)y = x$	(BQ2b)	dual
3-basis	$y(y\backslash x) = x$	(BQ3a)	projection
	$(x/y)y = x$	(BQ3b)	dual
	$(xy)z = x(yz)$	(BQ3c)	3-variable law
4-basis	$y(y\backslash x) = x$	(BQ4a)	$xy = x/y = x\backslash y = y - x$
	$(x/y)y = x$	(BQ4b)	dual
	$x/x = y\backslash y$	(BQ4c)	$xy = x/y = x\backslash y = -x - y$
	$((x(yz))y)u = x(y((zy)u))$	(BQ4d)	3-variable law

Table 5: Ternary Groups

	Axioms		Independence
2-basis	$m(m(x, m(y, z, u), y), z, u) = x$	(TG2a)	projection
	$m(u, z, m(y, m(u, z, y), x)) = x$	(TG2b)	dual
	$m(y, y, x) = x$	(TG3a)	projection
3-basis	$m(x, y, y) = x$	(TG3b)	dual
	$m(m(x, y, z), u, v) = m(x, y, m(z, u, v))$	(TG3c)	3-variable law
	$m(y, y, x) = x$	(TG4a)	4-model
4-basis	$m(x, y, y) = x$	(TG4b)	dual
	$m(m(m(x, y, z), z, u), u, y) = x$	(TG4c)	3-model
	$m(y, u, m(u, z, m(z, y, x))) = x$	(TG4d)	dual

Table 6: Group Theory Schema

	Axioms		Independence
2-basis	$((xy)(\alpha z))(y(\alpha z))' = x$	(GTS2a)	projection
	$((z\beta)y)'((z\beta)(yx)) = x$	(GTS2b)	dual
	$y(y'x) = x$	(GTS3a)	projection
3-basis	$(xy')y = x$	(GTS3b)	dual
	$((x\alpha)y)(\beta z) = (x\alpha)(y(\beta z))$	(GTS3c)	3-variable law
	$y(y'x) = x$	(GTS4a)	$\left\{ \begin{array}{l} xy = y - x \\ x' = x \\ \alpha = \beta = 0 \end{array} \right.$
4-basis	$(xy')y = x$	(GTS4b)	dual
	$x'x = yy'$	(GTS4c)	$\left\{ \begin{array}{l} xy = -x - y \\ x' = x \\ \alpha = \beta = 0 \end{array} \right.$
	$((x(\alpha y))\beta)z = x(\alpha((y\beta)z))$	(GTS4d)	3-variable law

Table 7: Symmetric Difference

	Axioms		Independence
2-basis	$(x\sharp(y\sharp z))\sharp((u\sharp y)\sharp(u\sharp z)) = x$	(SD2a)	projection
	$((z\sharp u)\sharp(y\sharp u))\sharp((z\sharp y)\sharp x) = x$	(SD2b)	dual
3-basis	$(y\sharp y)\sharp((z\sharp z)\sharp x) = x$	(SD3a)	projection
	$(x\sharp(z\sharp z))\sharp(y\sharp y) = x$	(SD3b)	dual
	$(x\sharp y)\sharp(z\sharp u) = (x\sharp z)\sharp(y\sharp u)$	(SD3c)	3-model
4-basis	$(x\sharp((z\sharp z)\sharp z))\sharp z = x$	(SD4a)	projection
	$z\sharp((z\sharp(z\sharp z))\sharp x) = x$	(SD4b)	dual
	$((x\sharp x)\sharp(x\sharp y))\sharp((y\sharp x)\sharp(x\sharp x)) = x\sharp x$	(SD4c)	$x\sharp y$=$2x$+$2y$+1(mod3)
	$(x\sharp y)\sharp(z\sharp u) = (x\sharp z)\sharp(y\sharp u)$	(SD4d)	3-variable law

Table 8: Symmetric Difference Schema

	Axioms		Independence
2-basis	$(x\sharp((y\sharp z)\sharp(y\sharp((z\sharp\alpha)\sharp v))))\sharp v = x$	(SDS2a)	projection
	$v\sharp((((v\sharp(\beta\sharp z))\sharp y)\sharp(z\sharp y))\sharp x) = x$	(SDS2b)	dual
3-basis	$(y\sharp y)\sharp(\alpha\sharp x) = x$	(SDS3a)	projection
	$(x\sharp\beta)\sharp(y\sharp y) = x$	(SDS3b)	dual
	$(x\sharp y)\sharp(z\sharp u) = (x\sharp z)\sharp(y\sharp u)$	(SDS3c)	3-model
4-basis	$(x\sharp((z\sharp z)\sharp z))\sharp z = x$	(SDS4a)	projection
	$z\sharp((z\sharp(z\sharp z))\sharp x) = x$	(SDS4b)	dual
	$((x\sharp x)\sharp(x\sharp y))\sharp((y\sharp x)\sharp(x\sharp x)) = \alpha$	(SDS4c)	$\begin{cases} x\sharp y{=}2x{+}2y{+}1(\mathrm{mod}3) \\ \alpha = 0 \end{cases}$
	$(x\sharp y)\sharp(z\sharp u) = (x\sharp z)\sharp(y\sharp u)$	(SDS4d)	10-model

Appendix

This appendix contains OTTER proofs that the axiom sets listed in Section 5 are indeed bases for the intended theories. The OTTER input files that produced these proofs can be found at `http://www.mcs.anl.gov/~mccune/papers/tarski`.

Boolean Groups (BG)

BG2

1 $x = x$ [axiom]

2	$((xy)z)(yz) = x$	[axiom]
3	$(zy)(z(yx)) = x$	[axiom]
4	$(AB)C \neq A(BC) \ \mid\ BB \neq AA \ \mid\ A(BB) \neq A$	[denial]
6	$x(y(zy)) = xz$	[2 (1) → 2 (1.1)]
15	$(x(yz))(xu) = y(zu)$	[3 (1) → 3 (1.2.2)]
26,25	$x(y(z(yx))) = z$	[3 (1) → 2 (1.1)]
29	$x((y(zy))(uz)) = xu$	[6 (1) → 6 (1.2.2)]
36	$(xy)y = x$	[3 (1) → 6 (1)]
43	$(xy)(x((z(yz))u)) = u$	[6 (1) → 3 (1.1)]
45	$((xy)z)((u(yu))z) = x$	[6 (1) → 2 (1.1.1)]
50,49	$x(y(zx)) = yz$	[3 (1) → 36 (1.1)]
52,51	$(xy)z = x(yz)$	[2 (1) → 36 (1.1)]
56,55	$x(yy) = x$	[45,52,52,52,52,52,26]
57	$x(y(x(z(y(zu))))) = u$	[43,52,52,52]
64,63	$x(y(yz)) = xz$	[29,52,52,50]
68,67	$x(y(z(xu))) = y(zu)$	[15,52,52]
73	$BB \neq AA$	[4,52,56,1,1]
76	$x(xy) = y$	[57,68,64]
79	$xx = yy$	[55 (1) → 76 (1.2)]
80	$\square$	[79.1,73.1]

BG3

1	$x = x$	[axiom]
2	$y(yx) = x$	[axiom]
3	$(xy)y = x$	[axiom]
4	$(xy)z = x(yz)$	[axiom]
5	$(AB)C \neq A(BC) \ \mid\ BB \neq AA \ \mid\ A(BB) \neq A$	[denial]
6	$x(yx) = y$	[2 (1) → 3 (1.1)]
10	$xy = yx$	[3 (1) → 6 (1.2)]
12,11	$(xy)z = y(xz)$	[10 (1) → 4 (1.1)]
22,21	$x(yy) = x$	[3 (1) → 4 (1)]
23	$B(AC) \neq A(BC) \ \mid\ BB \neq AA$	[5,12,22,1]
24	$x(yz) = y(xz)$	[4,12]
27	$BB \neq AA$	[24.1,23.1]
28	$xx = yy$	[21 (1) → 2 (1.2)]
29	$\square$	[28.1,27.1]

BG4

1	$x = x$	[axiom]
2	$y(yx) = x$	[axiom]
3	$(xy)y = x$	[axiom]
4	$xx = yy$	[axiom]
5	$(xy)(zu) = (xz)(yu)$	[axiom]
6	$(AB)C \neq A(BC) \mid BB \neq AA \mid A(BB) \neq A$	[denial]
11	$(xx)y = y$	[4 (1) → 3 (1.1)]
14,13	$x(yy) = x$	[4 (1) → 2 (1.2)]
15	$(AB)C \neq A(BC)$	[6,14,4,1]
53	$(xy)z = x(yz)$	[11 (1) → 5 (1.2),14]
55	$\square$	[53.1,15.1]

Abelian Groups (AG)

AG2

1	$x = x$	[axiom]
2	$(y(xz))(zy)' = x$	[axiom]
3	$(yz)'((zx)y) = x$	[axiom]
4	$BB' \neq AA' \mid A(BB') \neq A \mid (AB)C \neq A(BC) \mid BA \neq AB$	[denial]
5	$(xy)((zu)'x)' = u(yz)$	[2 (1) → 2 (1.1.2)]
14	$(x(y(zu)))'(zx) = (uy)'$	[2 (1) → 3 (1.2.1)]
15	$((xy)'y)'z = zx$	[2 (1) → 3 (1.2)]
18	$xy = ((yz)'z)'x$	[15]
21	$x(y(yz)')' = zx$	[3 (1) → 2 (1.1)]
29	$((xy)(zx)')z = y$	[3 (1) → 15 (1)]
90	$((((xy)'y)'z)(uz)')u = x$	[18 (1) → 29 (1.1.1)]
120	$x((yx)'(zy)) = z$	[2 (1) → 21 (1)]
131	$(x((yx)'z))y = z$	[21 (1) → 29 (1.1)]
224	$(x(y(zx)'))z = (u(uy)')'$	[21 (1) → 131 (1.1.2)]
225	$(x(((yz)'z)'(ux)'))u = y$	[18 (1) → 131 (1.1.2)]
236,235	$((xy)'y)' = x$	[2 (1) → 131 (1)]

237	$(x(xy)')' = (z(y(uz)'))u$	[224]
243,242	$(x(y(zx)'))z = y$	[225,236]
275	$((xy)(zy)')z = x$	[90,236]
288	$xy = yx$	[18,236]
290,289	$(x(xy)')' = y$	[237,243,flip.1]
291	$BB' \neq AA' \mid A(BB') \neq A \mid (AB)C \neq A(BC)$	[288.1,4.4]
359	$((xy)'x)' = y$	[288 (1) $\rightarrow$ 235 (1.1.1.1)]
361	$(x'y)' = z((yz)'x)$	[131 (1) $\rightarrow$ 235 (1.1.1.1)]
372	$x((yx)'z) = (z'y)'$	[361]
382,381	$(xy)z = x(yz)$	[235 (1) $\rightarrow$ 5 (1.2)]
440	$BB' \neq AA' \mid A(BB') \neq A$	[291,382,1]
445	$x(y((zy)'z)) = x$	[275,382,382,382]
531	$(x'y)' = (zy)'(xz)$	[120 (1) $\rightarrow$ 359 (1.1.1.1)]
533	$(xy)'(zx) = (z'y)'$	[531]
541	$x'(xy) = y$	[120 (1) $\rightarrow$ 14 (1.1.1),290]
578	$x'(yx) = y$	[288 (1) $\rightarrow$ 541 (1.2)]
581,580	$(xy)'(zx) = y'z$	[120 (1) $\rightarrow$ 541 (1.2)]
596,595	$(x'y)' = y'x$	[533,581]
607	$x(x'y) = y$	[120,581]
686,685	$x((yx)'z) = y'z$	[372,596]
704	$x(y'y) = x$	[445,686]
717,716	$x(yy') = x$	[288 (1) $\rightarrow$ 578 (1),382]
720	$BB' \neq AA'$	[440,717,1]
806	$B'B \neq AA'$	[288 (1) $\rightarrow$ 720 (1)]
827	$xx' = y'y$	[704 (1) $\rightarrow$ 607 (1.2)]
829	$x'x = yy'$	[827]
830	$\Box$	[829.1,806.1]

AG3

1	$x = x$	[axiom]
2	$y'(yx) = x$	[axiom]
3	$(xy)y' = x$	[axiom]
4	$x((y(uu'))v) = (v((w'w)x))y$	[axiom]
5	$BB' \neq AA' \mid A(BB') \neq A \mid (AB)C \neq A(BC) \mid BA \neq AB$	[denial]
6	$x''y = xy$	[2 (1) $\rightarrow$ 2 (1.2)]

11,10	$x(yx)' = y'$	[2 (1) $\rightarrow$ 3 (1.1)]
15,14	$x(x'y) = y$	[2 (1) $\rightarrow$ 6 (1)]
19,18	$x'' = x$	[14 (1) $\rightarrow$ 3 (1.1),11]
21,20	$(xy')y = x$	[18 (1) $\rightarrow$ 3 (1.2)]
23	$x(yz) = (z((u'u)x))y$	[14 (1) $\rightarrow$ 4 (1.2.1),19]
28	$(x((y'y)z))u = (x((v'v)z))u$	[4 (1) $\rightarrow$ 4 (1)]
32	$(x((y'y)z))u = z(ux)$	[23]
48,47	$(xy)' = y'x'$	[3 (1) $\rightarrow$ 10 (1.2.1)]
63	$(x(y'z'))(zy) = x$	[47 (1) $\rightarrow$ 20 (1.1.2)]
65	$(xy)((y'x')z) = z$	[47 (1) $\rightarrow$ 14 (1.2.1)]
69	$(x'y')((yx)z) = z$	[47 (1) $\rightarrow$ 2 (1.1)]
78,77	$(x(y'z))(z'y) = x$	[18 (1) $\rightarrow$ 63 (1.1.2.2)]
81	$(x((y'z)z'))y = x$	[14 (1) $\rightarrow$ 63 (1.2),48,19]
85	$x((y'(yx'))z) = z$	[20 (1) $\rightarrow$ 65 (1.1),48,19]
154	$x((yz)z') = xy$	[81 (1) $\rightarrow$ 81 (1.1),48,19,48,19,15]
174	$(x'((y'y)z))(ux) = zu$	[3 (1) $\rightarrow$ 23 (1.2)]
237,236	$(x'(xy))z = yz$	[85 (1) $\rightarrow$ 85 (1.2),48,48,19,19,21]
238	$((xy)y')z = xz$	[154 (1) $\rightarrow$ 236 (1.1.2),237]
258,257	$(x((y'y)z))u = (xz)u$	[20 (1) $\rightarrow$ 28 (1.1.2),19]
262	$((x'x)y)z = yz$	[69 (1) $\rightarrow$ 28 (1.1),19,258]
337,336	$(x'y)(zx) = yz$	[174,258]
375	$(xy)z = y(zx)$	[32,258]
451	$(xx')y = (z'z)y$	[262 (1) $\rightarrow$ 238 (1.1)]
454,453	$(x'x)y = y$	[262 (1) $\rightarrow$ 77 (1.1),78]
457	$(xx')y = y$	[451,454,flip.1]
461	$x(yx') = y$	[375 (1) $\rightarrow$ 453 (1)]
504	$xx' = yy'$	[457 (1) $\rightarrow$ 77 (1.1),337]
518	$A(BB') \neq A \mid (AB)C \neq A(BC) \mid BA \neq AB$	[504.1,5.1]
522	$xy = yx$	[20 (1) $\rightarrow$ 461 (1.2),19]
524,523	$x(yy') = x$	[457 (1) $\rightarrow$ 461 (1),48,19]
530	$(AB)C \neq A(BC)$	[522.1,518.3,524,1]
560	$x(yz) = y(zx)$	[375 (1) $\rightarrow$ 522 (1)]
603	$C(AB) \neq A(BC)$	[522 (1) $\rightarrow$ 530 (1)]
604	$\square$	[603.1,560.1]

AG4

1	$x = x$	[axiom]
2	$y'(yx) = x$	[axiom]
3	$(xy)y' = x$	[axiom]
4	$xx' = y'y$	[axiom]
5	$(xy)(zu) = (xz)(yu)$	[axiom]
6	$BB' \neq AA' \mid A(BB') \neq A \mid (AB)C \neq A(BC) \mid BA \neq AB$	[denial]
7	$x''y = xy$	[2 (1) → 2 (1.2)]
10,9	$xy'' = xy$	[3 (1) → 3 (1.1)]
12,11	$x(yx)' = y'$	[2 (1) → 3 (1.1)]
17,16	$(x'x)y = y$	[4 (1) → 3 (1.1),10]
25	$x(x'y) = y$	[2 (1) → 7 (1)]
31	$x(yz) = y(xz)$	[16 (1) → 5 (1.1),17]
38	$(xy)z = (x(u'u))(yz)$	[16 (1) → 5 (1.2)]
47	$(x(y'y))(zu) = (xz)u$	[38]
56	$x(y'y) = x''$	[4 (1) → 25 (1.2)]
59	$x'' = x(y'y)$	[56]
63,62	$x'' = x$	[25 (1) → 3 (1.1),12]
67,66	$x(y'y) = x$	[59,63]
71,70	$(xy)z = x(yz)$	[47,67]
92	$BB' \neq AA' \mid A(BB') \neq A \mid BA \neq AB$	[6,71,1]
95,94	$x(yy') = x$	[3,71]
96	$BB' \neq AA' \mid BA \neq AB$	[92,95,1]
105	$xx' = yy'$	[94 (1) → 25 (1.2)]
106	$BA \neq AB$	[105.1,96.1]
109	$xy = yx$	[94 (1) → 31 (1.2),95]
110	$\square$	[109.1,106.1]

Group Theory (GT)

GT2

1	$x = x$	[axiom]
2	$((xy)z)(yz)' = x$	[axiom]
3	$(zy)'(z(yx)) = x$	[axiom]
4	$BB' \neq AA' \mid A(BB') \neq A \mid (AB)C \neq A(BC)$	[denial]

5	$(xy)((zu)'y)' = (xz)u$	[2 (1) → 2 (1.1.1)]
6	$x(y(zy)')' = xz$	[2 (1) → 2 (1.1)]
8	$((x((yz)u))(zu)')y' = x$	[2 (1) → 2 (1.2.1)]
18,17	$(x((yz)u))'(xy) = (zu)'$	[2 (1) → 3 (1.2.2)]
30,29	$x((yz)'u')' = x((uy)z)$	[2 (1) → 6 (1.2.1.2.1)]
33	$((xy)(zy)')z = x$	[2 (1) → 6 (1)]
54	$((x(yz)')y)z = x$	[6 (1) → 33 (1.1)]
62	$(x(yx)')' = y$	[33 (1) → 3 (1.2.2),18]
75,74	$(((xy)z)u)v = x(((yz)u)v)$	[2 (1) → 5 (1.1),30]
106	$((xy)'x)' = y$	[62 (1) → 62 (1.1.2)]
110,109	$x(y((xy)'z)) = z$	[62 (1) → 3 (1.1)]
119	$x((yx)'(yz)) = z$	[106 (1) → 3 (1.1)]
206	$x((yy')z) = xz$	[8 (1) → 54 (1.1)]
257	$((x((yz)((uz)'(uv))))v')y' = x$	[119 (1) → 8 (1.1.2.1)]
284	$x(((yy')x)'z) = z$	[206 (1) → 119 (1.2)]
287,286	$(xx')y = y$	[206 (1) → 109 (1.2.2),110]
290	$x'(xy) = y$	[206 (1) → 3 (1),287]
295,294	$x(x'y) = y$	[284,287]
308	$xx' = yy'$	[286 (1) → 8 (1.1.1),75,287]
329	$A(BB') \neq A \mid (AB)C \neq A(BC)$	[308.1,4.1]
337,336	$x''y = xy$	[290 (1) → 290 (1.2)]
339,338	$(xy)'(xz) = y'z$	[119 (1) → 290 (1.2)]
350,349	$(xy)z = x(yz)$	[3 (1) → 290 (1.2),337]
358,357	$x(yy') = x$	[257,339,350,295,350,350,350,350,350,295]
370	$\square$	[329,358,350,1,1]

GT3

1	$x = x$	[axiom]
2	$y(y'x) = x$	[axiom]
3	$(xy')y = x$	[axiom]
4	$(xy)z = x(yz)$	[axiom]
5	$BB' \neq AA' \mid A(BB') \neq A \mid (AB)C \neq A(BC)$	[denial]
8	$xy'' = xy$	[3 (1) → 3 (1.1)]
18	$(xy)y' = x$	[8 (1) → 3 (1.1)]
39,38	$x(yy') = x$	[4 (1) → 18 (1)]
40	$BB' \neq AA'$	[5,39,1,4]

66	$xx' = yy'$	[38 (1) $\rightarrow$ 2 (1.2)]
67	$\square$	[66.1,40.1]

GT4

1	$x = x$	[axiom]
2	$y(y'x) = x$	[axiom]
3	$(xy')y = x$	[axiom]
4	$x'x = yy'$	[axiom]
5	$((x(uy))u)z = x(u((yu)z))$	[axiom]
6	$BB' \neq AA' \ \mid\ A(BB') \neq A \ \mid\ (AB)C \neq A(BC)$	[denial]
11	$xx' = yy'$	[4 (1) $\rightarrow$ 4 (1)]
12	$A(BB') \neq A \ \mid\ (AB)C \neq A(BC)$	[11.1,6.1]
13	$(xx')y = y''$	[4 (1) $\rightarrow$ 3 (1.1)]
15,14	$x(yy') = x$	[4 (1) $\rightarrow$ 2 (1.2)]
16	$x'' = (yy')x$	[13]
17	$(AB)C \neq A(BC)$	[12,15,1]
22,21	$x'' = x$	[2 (1) $\rightarrow$ 14 (1)]
25,24	$(xx')y = y$	[16,22]
29,28	$(xy')z = x(y'z)$	[4 (1) $\rightarrow$ 5 (1.1.1.2),15,25]
33,32	$(xy)'(x((yx)z)) = xz$	[4 (1) $\rightarrow$ 5 (1.1.1),25]
34	$(xy)z = x(yz)$	[3 (1) $\rightarrow$ 5 (1.1.1),29,33]
36	$\square$	[34.1,17.1]

Birkhoff Quasigroup (BQ)

BQ2

1	$x = x$	[axiom]
2	$y(z((yz)\backslash x)) = x$	[axiom]
3	$((x/(zy))z)y = x$	[axiom]
4	$B/B \neq A\backslash A \ \mid\ A(B/B) \neq A \ \mid\ (AB)C \neq A(BC)$	[denial]
5	$x((y((xy)\backslash z))(z\backslash u)) = u$	[2 (1) $\rightarrow$ 2 (1.2.2.1)]
7	$((x/y)((y/(zu))z))u = x$	[3 (1) $\rightarrow$ 3 (1.1.1.2)]
9	$((x/y)z)(u((zu)\backslash y)) = x$	[2 (1) $\rightarrow$ 3 (1.1.1.2)]
13	$x((((y((xy)\backslash z))(z\backslash u))(u\backslash v))(v\backslash w)) = w$	[5 (1) $\rightarrow$ 5 (1.2.1.2.1)]

15	$((x/(yz))y)((z(x\backslash u))(u\backslash v)) = v$	[3 (1) → 5 (1.2.1.2.1)]
17	$x(((y((xy)\backslash z))(z\backslash u))(u\backslash v)) = v$	[2 (1) → 5 (1.2.1.2.1)]
21	$((x/y)((y/z)((z/u)((u/(vw))v))))w = x$	[7 (1) → 7 (1.1.2.1.2)]
25	$((x/y)((y/z)((z/(uv))u)))v = x$	[3 (1) → 7 (1.1.2.1.2)]
29	$((x/y)((y/(zu))z))((u(x\backslash v))(v\backslash w)) = w$	[7 (1) → 5 (1.2.1.2.1)]
35	$x(y(((z(((u/v)z)\backslash x))y)\backslash v)) = u$	[2 (1) → 9 (1.1)]
43	$((x/y)((z/(uv))u))(v(z\backslash y)) = x$	[3 (1) → 9 (1.2.2.1)]
87	$((x/(y(z(u\backslash x))))y)z = u$	[15 (1) → 43 (1)]
100,99	$(x(((y/z)x)\backslash u))v = ((z/y)u)v$	[35 (1) → 87 (1.1.1.2)]
115	$((x/y)((y/x)z))u = zu$	[9 (1) → 87 (1.1.1.2)]
124,123	$((x/x)y)z = yz$	[2 (1) → 87 (1.1.1.2)]
152,151	$((x(x\backslash y))(y\backslash z))u = zu$	[17 (1) → 123 (1.1),100,124]
153	$(x(x\backslash y))z = yz$	[13 (1) → 123 (1.1),100,124,152]
235	$((x/(yx))y)(z(z\backslash u)) = u$	[153 (1) → 15 (1.2)]
322,321	$x(y(z\backslash z)) = xy$	[43 (1) → 115 (1)]
348,347	$x(y(y\backslash z)) = xz$	[153 (1) → 321 (1.2),322]
349	$x((y/y)z) = xz$	[123 (1) → 321 (1.2),322]
356,355	$x((y/z)((z/u)((u/(v(w\backslash w)))v))) = xy$	[25 (1) → 321 (1.2)]
357	$x((y/z)z) = xy$	[21 (1) → 321 (1.2),356]
366,365	$((x/(yx))y)z = z$	[235,348]
423	$((x/y)((y/(z(u/u)))z))((x\backslash v)(v\backslash w)) = w$	[123 (1) → 29 (1.2)]
437	$((x/y)\backslash z)u = ((y/x)z)u$	[347 (1) → 115 (1.1)]
439	$x(y/y) = x(z\backslash z)$	[321 (1) → 349 (1)]
452	$x(y\backslash y) = x(z/z)$	[439]
518,517	$(x/y)y = x$	[357 (1) → 365 (1),366]
520,519	$(x/x)y = y$	[349 (1) → 365 (1),366]
521	$x(x\backslash y) = y$	[347 (1) → 365 (1),366]
526,525	$x(y\backslash y) = x$	[321 (1) → 365 (1),366]
531	$x((yx)\backslash (yz)) = z$	[9 (1) → 365 (1)]
534,533	$x(y/y) = x$	[452,526]
541	$x((x\backslash y)(y\backslash z)) = z$	[423,534,518,518]
551	$A\backslash A \neq B/B \ \mid \ (AB)C \neq A(BC)$	[4,534,1]
662	$x\backslash x = y/y$	[519 (1) → 525 (1)]
666	$(x/y)((y/x)z) = z$	[115 (1) → 525 (1),526]
669	$(AB)C \neq A(BC)$	[662.1,551.1]
907	$x\backslash (xy) = y$	[533 (1) → 531 (1.2.1),520]
935,934	$x((yx)\backslash z) = y\backslash z$	[521 (1) → 531 (1.2.2)]
936	$(x/y)\backslash x = y$	[517 (1) → 531 (1.2.2),935]

1147	$(((xy)/(z(uy)))z)u = x$	[907 (1) → 87 (1.1.1.2.2.2)]
1166,1165	$((x/(y(zu)))y)z = x/u$	[936 (1) → 87 (1.1.1.2.2.2)]
1171	$(xy)/y = x$	[1147,1166]
1173	$x/(y\backslash x) = y$	[87,1166]
1180	$(x/(yz))y = x/z$	[3 (1) → 1171 (1.1)]
1386	$x((x\backslash y)z) = yz$	[907 (1) → 541 (1.2.2)]
1397,1396	$(x/y)\backslash z = (y/x)z$	[533 (1) → 437 (1),534]
1422	$x(((x\backslash y)/y)z) = z$	[1173 (1) → 666 (1.1)]
1650,1649	$x/(y\backslash z) = (x/z)y$	[521 (1) → 1180 (1.1.2)]
2213	$(xy)z = x(yz)$	[1386 (1) → 1422 (1.2),1397,1650,520]
2215	$\square$	[2213.1,669.1]

BQ3

1	$x = x$	[axiom]
2	$y(y\backslash x) = x$	[axiom]
3	$(x/y)y = x$	[axiom]
4	$(xy)z = x(yz)$	[axiom]
5	$B/B \neq A\backslash A \ \mid\ A(B/B) \neq A \ \mid\ (AB)C \neq A(BC)$	[denial]
8	$(x/y)(yz) = xz$	[3 (1) → 4 (1.1)]
20	$(x/y)z = x(y\backslash z)$	[2 (1) → 8 (1.2)]
21	$x(y\backslash z) = (x/y)z$	[20]
98,97	$x(y\backslash y) = x$	[3 (1) → 20 (1)]
160	$(x/x)y = y$	[2 (1) → 21 (1)]
179	$x/x = y\backslash y$	[97 (1) → 160 (1)]
186	$A(B/B) \neq A$	[179.1,5.1,4]
265	$A \neq A$	[179 (1) → 186 (1.2),98]
266	$\square$	[265.1,1.1]

BQ4

1	$x = x$	[axiom]
2	$y(y\backslash x) = x$	[axiom]
3	$(x/y)y = x$	[axiom]
4	$x/x = y\backslash y$	[axiom]
5	$((x(yz))y)u = x(y((zy)u))$	[axiom]

6	$B/B \neq A\backslash A \ \mid\ A(B/B) \neq A \ \mid\ (AB)C \neq A(BC)$	[denial]
7	$x\backslash x = y\backslash y$	[4 (1) → 4 (1)]
9,8	$(x\backslash x)y = y$	[4 (1) → 3 (1.1)]
11,10	$x(y\backslash y) = x$	[7 (1) → 2 (1.2)]
15,14	$(xy)z = x(yz)$	[8 (1) → 5 (1.1.1.2),11,11,9]
24	$A\backslash A \neq B/B \ \mid\ A(B/B) \neq A$	[6,15,1]
27	$(x/y)(yz) = xz$	[3 (1) → 14 (1.1)]
39	$(x/y)z = x(y\backslash z)$	[2 (1) → 27 (1.2)]
40	$x(y\backslash z) = (x/y)z$	[39]
100	$(x/x)y = y$	[2 (1) → 40 (1)]
116	$(x/x)\backslash y = y$	[2 (1) → 100 (1)]
123	$x\backslash x = y/y$	[100 (1) → 10 (1)]
124	$A(B/B) \neq A$	[123.1,24.1]
127	$x(y/y) = x$	[116 (1) → 10 (1.2)]
129	$\square$	[127.1,124.1]

Ternary Group (TG)

TG2

1	$x = x$	[axiom]
2	$m(m(x, m(y, z, u), y), z, u) = x$	[axiom]
3	$m(u, z, m(y, m(u, z, y), x)) = x$	[axiom]
4	$m(m(A, B, C), D, F) \neq m(A, B, m(C, D, F)) \ \mid$ $m(A, A, B) \neq B \ \mid\ m(B, A, A) \neq B$	[denial]
5	$m(m(x, y, m(y, m(z, u, v), z)), u, v) = x$	[2 (1) → 2 (1.1.2)]
25	$m(x, y, z) = m(x, u, m(u, m(v, w, m(w, m(v6, y, z), v6)), v))$	[5 (1) → 5 (1.1)]
27,26	$m(x, y, y) = x$	[3 (1) → 5 (1.1)]
29,28	$m(x, m(y, z, m(z, m(u, v, w), u)), y) = m(x, v, w)$	[2 (1) → 5 (1.1)]
30	$m(x, y, m(y, z, u)) = m(x, z, u)$	[25,29]
31	$m(m(A, B, C), D, F) \neq m(A, B, m(C, D, F)) \ \mid$ $m(A, A, B) \neq B$	[4,27,1]
35	$m(x, y, m(y, x, z)) = z$	[26 (1) → 3 (1.3.2)]
53,52	$m(x, x, y) = y$	[5 (1) → 35 (1.3),29]
55	$m(m(A, B, C), D, F) \neq m(A, B, m(C, D, F))$	[31,53,1]
80	$m(m(x, y, z), u, v) = m(x, y, m(z, u, v))$	[30 (1) → 3 (1.3)]

82	$\square$	[80.1,55.1]

TG3

2	$m(y,y,x) = x$	[axiom]
3	$m(x,y,y) = x$	[axiom]
4	$m(m(x,y,z),u,v) = m(x,y,m(z,u,v))$	[axiom]
5	$m(m(A,B,C),D,F) \neq m(A,B,m(C,D,F)) \ \mid$ $m(A,A,B) \neq B \ \mid \ m(B,A,A) \neq B$	[denial]
6	$\square$	[4,5,2,3]

TG4

1	$x = x$	[axiom]
2	$m(y,y,x) = x$	[axiom]
3	$m(x,y,y) = x$	[axiom]
4	$m(m(m(x,y,z),z,u),u,y) = x$	[axiom]
5	$m(y,u,m(u,z,m(z,y,x))) = x$	[axiom]
6	$m(m(A,B,C),D,F) \neq m(A,B,m(C,D,F)) \ \mid$ $m(A,A,B) \neq B \ \mid \ m(B,A,A) \neq B$	[denial]
8	$m(m(x,y,z),z,y) = x$	[3 (1) $\rightarrow$ 4 (1.1.1)]
12	$m(m(x,y,z),z,u) = m(x,y,u)$	[4 (1) $\rightarrow$ 8 (1.1)]
17	$m(x,y,m(y,x,z)) = z$	[2 (1) $\rightarrow$ 5 (1.3.3)]
20	$m(x,y,z) = m(x,u,m(u,y,z))$	[5 (1) $\rightarrow$ 17 (1.3)]
21	$m(x,y,m(y,z,u)) = m(x,z,u)$	[20]
87,86	$m(m(x,y,z),u,v) = m(x,y,m(z,u,v))$	[12 (1) $\rightarrow$ 21 (1)]
90	$\square$	[6,87,1,2,3]

Group Theory Schema (GTS)

GTS2

1	$x = x$	[axiom]
2	$((xy)(\alpha z))(y(\alpha z))' = x$	[axiom]
3	$((z\beta)y)'((z\beta)(yx)) = x$	[axiom]
4	$BB' \neq AA' \ \mid \ A(BB') \neq A \ \mid \ (AB)C \neq A(BC)$	[denial]

5	$(x(\alpha y))((z(\alpha u))'(\alpha y))' = (xz)(\alpha u)$	[2 (1) → 2 (1.1.1)]
14,13	$((x(y(\alpha z))')y)(\alpha z) = x$	[2 (1) → 5 (1)]
22,21	$(((x(\beta(\alpha y))')\beta)\alpha)'x = y$	[13 (1) → 3 (1.2)]
31	$(xy)'(x(yz)) = z$	[21 (1) → 3 (1.2.1),22]
47	$((xy)'x)'z = yz$	[31 (1) → 31 (1.2)]
72,71	$x((yx)'(yz)) = z$	[31 (1) → 47 (1)]
73	$((xy)'x)' = y$	[47 (1) → 13 (1.1.1),14]
77	$(x'(yz)')' = y(zx)$	[31 (1) → 73 (1.1.1.1)]
83	$(x(yx)')' = y$	[73 (1) → 73 (1.1.1)]
109	$x(y((xy)'z)) = z$	[83 (1) → 31 (1.1)]
130	$((x(yz)')y)z = x$	[71 (1) → 13 (1.2),72]
140	$((xy)z)(yz)' = x$	[71 (1) → 2 (1.2.1.2),72]
163	$((xy)(zy)')z = x$	[73 (1) → 130 (1.1.1.2)]
264	$((xy)(zy)')(z(x'u)) = u$	[163 (1) → 109 (1.2.2.1.1)]
282,281	$x(y'y) = x$	[83 (1) → 77 (1)]
301	$(xy')y = x$	[130 (1) → 281 (1),282]
303	$(xy)y' = x$	[281 (1) → 140 (1.1),282]
407,406	$(xy)z = x(yz)$	[140 (1) → 301 (1.1)]
465,464	$x(yy') = x$	[303,407]
475	$x(x'y) = y$	[264,407,407,407,72]
499	$BB' \neq AA'$	[4,465,407,1,1]
513	$xx' = yy'$	[464 (1) → 475 (1.2)]
514	$\square$	[513.1,499.1]

GTS3

1	$x = x$	[axiom]
2	$y(y'x) = x$	[axiom]
3	$(xy')y = x$	[axiom]
4	$((x\alpha)y)(\beta z) = (x\alpha)(y(\beta z))$	[axiom]
5	$BB' \neq AA' \ \mid\ A(BB') \neq A \ \mid\ (AB)C \neq A(BC)$	[denial]
6	$x''y = xy$	[2 (1) → 2 (1.2)]
8	$xy'' = xy$	[3 (1) → 3 (1.1)]
12,11	$x'(xy) = y$	[6 (1) → 2 (1.2)]
17,16	$x'' = x$	[8 (1) → 11 (1.2),12]
19,18	$(xy)y' = x$	[8 (1) → 3 (1.1)]
20	$(xy)(\beta z) = ((x\alpha')\alpha)(y(\beta z))$	[3 (1) → 4 (1.1.1)]

30	$x(yx)' = y'$	[11 (1) → 18 (1.1)]
32	$((x\alpha)(y(\beta z)))(\beta z)' = (x\alpha)y$	[4 (1) → 18 (1.1)]
39,38	$(xy)' = y'x'$	[18 (1) → 30 (1.2.1)]
41,40	$((x\alpha)(y(\beta z)))(z'\beta') = (x\alpha)y$	[32,39]
48	$(x(yz))(z'y') = x$	[38 (1) → 18 (1.2)]
56	$(xy)(z(z'y')) = x$	[18 (1) → 48 (1.1.2),17,39]
104	$(xy)z = ((x\alpha')\alpha)(y(\beta(\beta'z)))$	[2 (1) → 20 (1.2)]
111,110	$((x\alpha')\alpha)y = xy$	[20 (1) → 48 (1.1),41]
129	$(xy)z = x(y(\beta(\beta'z)))$	[104,111]
183,182	$x(y(y'z)) = xz$	[56 (1) → 56 (1.1),39,39,17,17,19]
185,184	$(xy)z = x(yz)$	[129,183]
189,188	$x(yy') = x$	[56,183,185]
194	$BB' \neq AA'$	[5,189,185,1,1]
199	$xx' = yy'$	[188 (1) → 2 (1.2)]
200	$\square$	[199.1,194.1]

GTS4

1	$x = x$	[axiom]
2	$y(y'x) = x$	[axiom]
3	$(xy')y = x$	[axiom]
4	$x'x = yy'$	[axiom]
5	$((x(\alpha y))\beta)z = x(\alpha((y\beta)z))$	[axiom]
6	$BB' \neq AA' \mid A(BB') \neq A \mid (AB)C \neq A(BC)$	[denial]
11	$xx' = yy'$	[4 (1) → 4 (1)]
12	$A(BB') \neq A \mid (AB)C \neq A(BC)$	[11.1,6.1]
13	$(xx')y = y''$	[4 (1) → 3 (1.1)]
15,14	$x(yy') = x$	[4 (1) → 2 (1.2)]
16	$x'' = (yy')x$	[13]
17	$(AB)C \neq A(BC)$	[12,15,1]
22,21	$x'' = x$	[2 (1) → 14 (1)]
25,24	$(xx')y = y$	[16,22]
28	$((xy)\beta)z = x(\alpha(((\alpha'y)\beta)z))$	[2 (1) → 5 (1.1.1.2)]
29	$(x\beta)y = x(\alpha((\alpha'\beta)y))$	[14 (1) → 5 (1.1.1)]
39	$x(\alpha((\alpha'\beta)y)) = (x\beta)y$	[29]
44	$(xy)y' = x$	[21 (1) → 3 (1.1.2)]
46	$x'(xy) = y$	[21 (1) → 2 (1.2.1)]

49,48	$(x'x)y = y$	[21 (1) → 24 (1.1.2)]
59	$(xy)'x = y'$	[44 (1) → 46 (1.2)]
74,73	$(xy)' = y'x'$	[46 (1) → 59 (1.1.1)]
105	$(x'y')((yx)z) = z$	[73 (1) → 46 (1.1)]
160,159	$\alpha(((\alpha'x)\beta)y) = (x\beta)y$	[48 (1) → 28 (1.1.1),49]
167,166	$x((x'\beta)y) = \beta y$	[11 (1) → 28 (1.1.1),25,160]
173,172	$x(((x'y)\beta)z) = (y\beta)z$	[2 (1) → 28 (1.1.1),160]
183	$x((y\beta)((\beta'(y'x'))z)) = z$	[2 (1) → 28 (1),74,74,173]
195,194	$(x\beta)y = x(\beta y)$	[39,167]
215	$x(y(\beta((\beta'(y'x'))z))) = z$	[183,195]
237	$x(\beta((\beta'x')y)) = y$	[2 (1) → 194 (1),74]
423	$(x'y)((y'x)z) = z$	[21 (1) → 105 (1.1.2)]
502,501	$\beta((\beta'x)y) = xy$	[237 (1) → 2 (1.2),22]
505	$x(y((y'x')z)) = z$	[215,502]
695	$(xy)z = x(yz)$	[423 (1) → 505 (1.2.2),22]
697	$\square$	[695.1,17.1]

Symmetric Difference (SD)

SD2

2	$(x\sharp(y\sharp z))\sharp((u\sharp y)\sharp(u\sharp z)) = x$	[axiom]
3	$((z\sharp u)\sharp(y\sharp u))\sharp((z\sharp y)\sharp x) = x$	[axiom]
4	$(A\sharp B)\sharp(((A\sharp C)\sharp(D\sharp D))\sharp B) \neq C$	[denial]
13	$(x\sharp((y\sharp z)\sharp(u\sharp z)))\sharp(u\sharp y) = x$	[2 (1) → 2 (1.2)]
25	$x\sharp((x\sharp(y\sharp y))\sharp z) = z$	[2 (1) → 3 (1.1)]
469	$x\sharp((x\sharp(y\sharp z))\sharp z) = y$	[3 (1) → 13 (1)]
530	$(x\sharp((x\sharp y)\sharp z))\sharp z = y$	[3 (1) → 469 (1.2)]
598	$(x\sharp y)\sharp(((x\sharp z)\sharp(u\sharp u))\sharp y) = z$	[25 (1) → 530 (1.1.2)]
600	$\square$	[598.1,4.1]

SD3

2	$(y\sharp y)\sharp((z\sharp z)\sharp x) = x$	[axiom]
3	$(x\sharp(z\sharp z))\sharp(y\sharp y) = x$	[axiom]
4	$(x\sharp y)\sharp(z\sharp u) = (x\sharp z)\sharp(y\sharp u)$	[axiom]

5	$(A\sharp B)\sharp(((A\sharp C)\sharp(D\sharp D))\sharp B) \neq C$	[denial]
9	$x\sharp x = y\sharp y$	[2 (1) → 3 (1)]
28	$(x\sharp x)\sharp(y\sharp z) = (u\sharp y)\sharp(u\sharp z)$	[9 (1) → 4 (1.1)]
35	$(x\sharp y)\sharp(z\sharp z) = (x\sharp u)\sharp(y\sharp u)$	[9 (1) → 4 (1.2)]
41	$(x\sharp y)\sharp((z\sharp z)\sharp y) = x$	[3 (1) → 4 (1)]
43	$(x\sharp(y\sharp y))\sharp(x\sharp z) = z$	[2 (1) → 4 (1)]
45	$(x\sharp y)\sharp(x\sharp z) = (u\sharp u)\sharp(y\sharp z)$	[28]
48	$(x\sharp y)\sharp(z\sharp y) = (x\sharp z)\sharp(u\sharp u)$	[35]
70	$(x\sharp((y\sharp y)\sharp z))\sharp z = x$	[41 (1) → 41 (1.2)]
80	$x\sharp((x\sharp(y\sharp y))\sharp z) = z$	[43 (1) → 43 (1.1)]
122	$(x\sharp y)\sharp(((z\sharp z)\sharp(y\sharp u))\sharp u) = x$	[4 (1) → 70 (1)]
273,272	$(x\sharp x)\sharp(((y\sharp(z\sharp z))\sharp u)\sharp v) = u\sharp(y\sharp v)$	[80 (1) → 45 (1.1)]
1740	$x\sharp((x\sharp(y\sharp z))\sharp z) = y$	[80 (1) → 122 (1.1),273]
1875	$(x\sharp y)\sharp(((x\sharp z)\sharp(u\sharp u))\sharp y) = z$	[48 (1) → 1740 (1.2.1)]
1877	$\square$	[1875.1,5.1]

SD4

2	$(x\sharp((z\sharp z)\sharp z))\sharp z = x$	[axiom]
3	$z\sharp((z\sharp(z\sharp z))\sharp x) = x$	[axiom]
4	$((x\sharp x)\sharp(x\sharp y))\sharp((y\sharp x)\sharp(x\sharp x)) = x\sharp x$	[axiom]
5	$(x\sharp y)\sharp(z\sharp u) = (x\sharp z)\sharp(y\sharp u)$	[axiom]
6	$(A\sharp B)\sharp(((A\sharp C)\sharp(D\sharp D))\sharp B) \neq C$	[denial]
18	$(x\sharp((x\sharp y)\sharp((x\sharp y)\sharp(x\sharp y))))\sharp(y\sharp z) = z$	[3 (1) → 5 (1)]
20	$(x\sharp y)\sharp((((y\sharp z)\sharp(y\sharp z))\sharp(y\sharp z))\sharp z) = x$	[2 (1) → 5 (1)]
27	$x\sharp((x\sharp y)\sharp((x\sharp x)\sharp z)) = y\sharp z$	[5 (1) → 3 (1.2)]
33	$((x\sharp(y\sharp y))\sharp(z\sharp y))\sharp y = x\sharp z$	[5 (1) → 2 (1.1)]
46,45	$x\sharp((x\sharp y)\sharp((x\sharp z)\sharp(x\sharp u))) = y\sharp(z\sharp u)$	[5 (1) → 27 (1.2.2)]
50	$((((x\sharp x)\sharp y)\sharp((x\sharp x)\sharp y))\sharp((x\sharp x)\sharp y))\sharp y = x\sharp x$	[2 (1) → 27 (1.2)]
58	$(x\sharp(x\sharp x))\sharp(x\sharp y) = y$	[18,46]
72,71	$(x\sharp x)\sharp((x\sharp x)\sharp y) = y$	[5 (1) → 58 (1)]
96,95	$(x\sharp(y\sharp y))\sharp((y\sharp y)\sharp(y\sharp y)) = x$	[4 (1) → 2 (1.1.2.1),72]
109	$(x\sharp x)\sharp(y\sharp(((x\sharp x)\sharp(x\sharp x))\sharp z)) = ((x\sharp x)\sharp y)\sharp z$	[71 (1) → 27 (1.2.1)]
112	$(((x\sharp x)\sharp(x\sharp x))\sharp y)\sharp((x\sharp x)\sharp y) = (x\sharp x)\sharp(x\sharp x)$	[71 (1) → 4 (1.1.2),96]
114	$((x\sharp x)\sharp y)\sharp z = (x\sharp x)\sharp(y\sharp(((x\sharp x)\sharp(x\sharp x))\sharp z))$	[109]
122,121	$((x\sharp x)\sharp(y\sharp(x\sharp x)))\sharp(x\sharp x) = (x\sharp x)\sharp y$	[71 (1) → 33 (1.1.1)]
125,124	$(((x\sharp y)\sharp(z\sharp y))\sharp(u\sharp y))\sharp y = (x\sharp z)\sharp u$	[5 (1) → 33 (1.1.1)]

127,126	$((x\sharp x)\sharp(x\sharp x))\sharp y = (x\sharp x)\sharp y$	[4 (1) $\rightarrow$ 33 (1.1.1),122]
141,140	$(x\sharp x)\sharp(x\sharp x) = x\sharp x$	[50,125,127]
142	$(x\sharp y)\sharp((y\sharp y)\sharp y) = x$	[20,125]
145,144	$((x\sharp x)\sharp y)\sharp z = (x\sharp x)\sharp(y\sharp((x\sharp x)\sharp z))$	[114,141]
147,146	$(x\sharp x)\sharp(y\sharp y) = x\sharp x$	[112,141,145,72,141]
155	$(x\sharp(y\sharp y))\sharp(y\sharp y) = x$	[95,141]
175	$x\sharp x = y\sharp y$	[71 (1) $\rightarrow$ 142 (1.1),145,72,147,72]
194	$x\sharp((x\sharp y)\sharp(z\sharp z)) = y\sharp(x\sharp x)$	[175 (1) $\rightarrow$ 27 (1.2.2)]
210	$(x\sharp y)\sharp(z\sharp z) = (x\sharp u)\sharp(y\sharp u)$	[175 (1) $\rightarrow$ 5 (1.2)]
235,234	$(x\sharp(y\sharp y))\sharp(z\sharp z) = x$	[175 (1) $\rightarrow$ 155 (1.1.2)]
638	$(x\sharp y)\sharp(((x\sharp z)\sharp(u\sharp u))\sharp y) = z$	[194 (1) $\rightarrow$ 210 (1.1),235]
640	$\Box$	[638.1,6.1]

Symmetric Difference Schema (SDS)

SDS2

1	$x = x$	[axiom]
2	$(x\sharp((y\sharp z)\sharp(y\sharp((z\sharp\alpha)\sharp v))))\sharp v = x$	[axiom]
3	$v\sharp((((v\sharp(\beta\sharp z))\sharp y)\sharp(z\sharp y))\sharp x) = x$	[axiom]
4	$(A\sharp B)\sharp(((A\sharp C)\sharp\alpha)\sharp B) \neq C \mid \beta \neq \alpha$	[denial]
5	$x\sharp((x\sharp(\beta\sharp(y\sharp z)))\sharp(y\sharp((z\sharp\alpha)\sharp u))) = u$	[2 (1) $\rightarrow$ 3 (1.2)]
7	$(((x\sharp(\beta\sharp y))\sharp z)\sharp(((y\sharp z)\sharp\alpha)\sharp u))\sharp u = x$	[3 (1) $\rightarrow$ 2 (1.1)]
9	$x\sharp((x\sharp y)\sharp((((\beta\sharp(\beta\sharp z))\sharp u)\sharp(z\sharp u))\sharp((y\sharp\alpha)\sharp v))) = v$	[3 (1) $\rightarrow$ 5 (1.2.1.2)]
13	$x\sharp x = ((\beta\sharp(\beta\sharp y))\sharp z)\sharp(y\sharp z)$	[9 (1) $\rightarrow$ 2 (1.1)]
14	$((\beta\sharp(\beta\sharp x))\sharp y)\sharp(x\sharp y) = z\sharp z$	[13]
46	$x\sharp x = y\sharp y$	[14 (1) $\rightarrow$ 14 (1)]
60,59	$((x\sharp x)\sharp(((y\sharp(z\sharp(\beta\sharp y)))\sharp\alpha)\sharp u))\sharp u = \beta\sharp(\beta\sharp z)$	[14 (1) $\rightarrow$ 7 (1.1.1)]
63	$x\sharp((x\sharp(\beta\sharp(((\beta\sharp(\beta\sharp(y\sharp\alpha)))\sharp z)\sharp y)))\sharp(u\sharp u)) = z$	[14 (1) $\rightarrow$ 5 (1.2.2)]
77	$(x\sharp x)\sharp y = (\beta\sharp(\beta\sharp(z\sharp u)))\sharp(z\sharp((u\sharp\alpha)\sharp y))$	[14 (1) $\rightarrow$ 2 (1.1)]
84	$(\beta\sharp(\beta\sharp(x\sharp y)))\sharp(x\sharp((y\sharp\alpha)\sharp z)) = (u\sharp u)\sharp z$	[77]
87	$x\sharp((x\sharp(\beta\sharp(\beta\sharp y)))\sharp(z\sharp z)) = y\sharp\alpha$	[46 (1) $\rightarrow$ 9 (1.2.2)]
89	$x\sharp((x\sharp y)\sharp((((\beta\sharp(\beta\sharp z))\sharp u)\sharp(z\sharp u))\sharp(v\sharp v))) = y\sharp\alpha$	[46 (1) $\rightarrow$ 9 (1.2.2.2)]
91	$x\sharp((x\sharp\alpha)\sharp((((\beta\sharp(\beta\sharp y))\sharp z)\sharp(y\sharp z))\sharp((u\sharp u)\sharp v))) = v$	[46 (1) $\rightarrow$ 9 (1.2.2.2.1)]

97	$x\sharp((x\sharp y)\sharp((((\beta\sharp(z\sharp z))\sharp u)\sharp(\beta\sharp u))\sharp((y\sharp\alpha)\sharp v))) = v$	
		[46 (1) → 9 (1.2.2.1.1.1.2)]
99	$((\beta\sharp(\beta\sharp x))\sharp x)\sharp(y\sharp y) = z\sharp z$	[46 (1) → 14 (1.2)]
105,104	$\beta\sharp(\beta\sharp x) = x$	[46 (1) → 7 (1.1.1),60]
120	$x\sharp(y\sharp y) = ((x\sharp(\beta\sharp z))\sharp u)\sharp(z\sharp u)$	[46 (1) → 3 (1.2)]
142	$(x\sharp x)\sharp(y\sharp y) = z\sharp z$	[99,105]
145	$x\sharp((x\sharp\alpha)\sharp(((y\sharp z)\sharp(y\sharp z))\sharp((u\sharp u)\sharp v))) = v$	[91,105]
147	$x\sharp((x\sharp y)\sharp(((z\sharp u)\sharp(z\sharp u))\sharp(v\sharp v))) = y\sharp\alpha$	[89,105]
150,149	$x\sharp((x\sharp y)\sharp(z\sharp z)) = y\sharp\alpha$	[87,105]
151	$(x\sharp y)\sharp(x\sharp((y\sharp\alpha)\sharp z)) = (u\sharp u)\sharp z$	[84,105]
152	$(\beta\sharp(((x\sharp\alpha)\sharp y)\sharp x))\sharp\alpha = y$	[63,105,150]
167	$((x\sharp(\beta\sharp y))\sharp z)\sharp(y\sharp z) = x\sharp(u\sharp u)$	[120]
169,168	$\beta\sharp(x\sharp x) = \beta$	[46 (1) → 104 (1.2)]
171,170	$(x\sharp y)\sharp(x\sharp((y\sharp\alpha)\sharp z)) = \beta\sharp z$	[5 (1) → 104 (1.2),105]
173,172	$((x\sharp y)\sharp(x\sharp y))\sharp z = \beta\sharp z$	[3 (1) → 104 (1.2),105]
176	$x\sharp((x\sharp y)\sharp(\beta\sharp((y\sharp\alpha)\sharp z))) = z$	[97,169,173]
181,180	$(x\sharp x)\sharp y = \beta\sharp y$	[151,171]
184	$(x\sharp(\beta\sharp y))\sharp y = x$	[2,171]
186	$x\sharp((x\sharp y)\sharp\beta) = y\sharp\alpha$	[147,181,169]
188	$x\sharp((x\sharp\alpha)\sharp y) = y$	[145,181,181,105]
195,194	$x\sharp x = \beta$	[142,181,169]
210	$((x\sharp(\beta\sharp y))\sharp z)\sharp(y\sharp z) = x\sharp\beta$	[167,195]
232	$(\beta\sharp((x\sharp y)\sharp(((x\sharp(\beta\sharp z))\sharp u)\sharp(((z\sharp u)\sharp\alpha)\sharp\alpha))))\sharp\alpha = y$	
		[7 (1) → 152 (1.1.2.1.1)]
234	$(\beta\sharp(x\sharp y))\sharp\alpha = ((y\sharp\alpha)\sharp(\beta\sharp(z\sharp u)))\sharp(z\sharp((u\sharp\alpha)\sharp x))$	
		[5 (1) → 152 (1.1.2.1)]
236	$((x\sharp\alpha)\sharp(\beta\sharp(y\sharp z)))\sharp(y\sharp((z\sharp\alpha)\sharp u)) = (\beta\sharp(u\sharp x))\sharp\alpha$	[234]
248	$(x\sharp\beta)\sharp\beta = x$	[194 (1) → 184 (1.1.2)]
250	$(x\sharp y)\sharp(\beta\sharp y) = x$	[104 (1) → 184 (1.1.2)]
254	$(\beta\sharp(x\sharp\alpha))\sharp\alpha = \beta\sharp x$	[184 (1) → 152 (1.1.2)]
258	$((x\sharp y)\sharp(((z\sharp y)\sharp\alpha)\sharp u))\sharp u = x\sharp z$	[184 (1) → 7 (1.1.1.1),105]
272	$\alpha\sharp(\beta\sharp x) = x$	[194 (1) → 188 (1.2.1)]
279,278	$x\sharp\beta = x\sharp\alpha$	[194 (1) → 188 (1.2)]
283,282	$((x\sharp\alpha)\sharp(\beta\sharp(y\sharp z)))\sharp(y\sharp((z\sharp\alpha)\sharp u)) = x\sharp u$	[5 (1) → 188 (1.2)]
287,286	$(x\sharp\alpha)\sharp\alpha = x$	[248,279,279]
301,300	$((x\sharp(\beta\sharp y))\sharp z)\sharp(y\sharp z) = x\sharp\alpha$	[210,279]
302	$x\sharp((x\sharp y)\sharp\alpha) = y\sharp\alpha$	[186,279]
307,306	$(\beta\sharp(x\sharp y))\sharp\alpha = y\sharp x$	[236,283]

308	$(x\sharp\alpha)\sharp(x\sharp y) = y$	[232,287,301,307]
311,310	$\beta\sharp x = \alpha\sharp x$	[254,307]
322	$((x\sharp(\alpha\sharp y))\sharp z)\sharp(y\sharp z) = x\sharp\alpha$	[300,311]
337,336	$\alpha\sharp(\alpha\sharp x) = x$	[272,311]
348	$(x\sharp y)\sharp(\alpha\sharp y) = x$	[250,311]
364	$x\sharp((x\sharp y)\sharp(\alpha\sharp((y\sharp\alpha)\sharp z))) = z$	[176,311]
387,386	$\beta = \alpha$	[194 (1) $\rightarrow$ 286 (1.1),311,195]
389,388	$x\sharp x = \alpha$	[194,387]
390	$(A\sharp B)\sharp(((A\sharp C)\sharp\alpha)\sharp B) \neq C$	[4,387,1]
395	$(\alpha\sharp(((x\sharp y)\sharp\alpha)\sharp z))\sharp z = y\sharp x$	[388 (1) $\rightarrow$ 258 (1.1.1)]
398,397	$(x\sharp(y\sharp z))\sharp z = (x\sharp\alpha)\sharp y$	[286 (1) $\rightarrow$ 258 (1.1.1),287]
409	$((x\sharp y)\sharp(z\sharp y))\sharp\alpha = x\sharp z$	[286 (1) $\rightarrow$ 258 (1.1.2)]
413	$((x\sharp y)\sharp\alpha)\sharp x = \alpha\sharp y$	[388 (1) $\rightarrow$ 258 (1.1)]
419	$\alpha\sharp((x\sharp y)\sharp\alpha) = y\sharp x$	[395,398,389]
438	$x\sharp(\alpha\sharp(y\sharp x)) = y\sharp\alpha$	[308 (1) $\rightarrow$ 348 (1.1)]
441	$x\sharp\alpha = y\sharp(\alpha\sharp(x\sharp y))$	[438]
470	$(x\sharp y)\sharp(x\sharp\alpha) = (\alpha\sharp y)\sharp\alpha$	[348 (1) $\rightarrow$ 302 (1.2.1)]
472,471	$(\alpha\sharp x)\sharp\alpha = \alpha\sharp(x\sharp\alpha)$	[336 (1) $\rightarrow$ 302 (1.2.1)]
476	$(x\sharp y)\sharp(x\sharp\alpha) = \alpha\sharp(y\sharp\alpha)$	[470,472,flip.1]
497	$(\alpha\sharp x)\sharp(\alpha\sharp y) = (y\sharp x)\sharp\alpha$	[413 (1) $\rightarrow$ 348 (1.1)]
500	$(x\sharp y)\sharp\alpha = (\alpha\sharp y)\sharp(\alpha\sharp x)$	[497]
504,503	$(x\sharp y)\sharp\alpha = \alpha\sharp(y\sharp x)$	[419 (1) $\rightarrow$ 336 (1.2)]
506,505	$(\alpha\sharp x)\sharp(\alpha\sharp y) = \alpha\sharp(x\sharp y)$	[500,504]
523	$\alpha\sharp((x\sharp y)\sharp(z\sharp y)) = z\sharp x$	[409,504]
531	$(A\sharp B)\sharp((\alpha\sharp(C\sharp A))\sharp B) \neq C$	[390,504]
535	$x\sharp y = \alpha\sharp((y\sharp z)\sharp(x\sharp z))$	[523]
551,550	$(x\sharp(\alpha\sharp y))\sharp z = (x\sharp\alpha)\sharp(\alpha\sharp(y\sharp z))$	[322 (1) $\rightarrow$ 348 (1.1)]
566,565	$x\sharp(\alpha\sharp((y\sharp z)\sharp x)) = \alpha\sharp(z\sharp y)$	[441 (1) $\rightarrow$ 503 (1)]
668,667	$(x\sharp y)\sharp(\alpha\sharp((y\sharp\alpha)\sharp(x\sharp z))) = z$	[476 (1) $\rightarrow$ 364 (1.2.1),504,337,506]
735,734	$(\alpha\sharp x)\sharp y = \alpha\sharp(x\sharp(\alpha\sharp y))$	[336 (1) $\rightarrow$ 505 (1.2)]
785	$(A\sharp B)\sharp(\alpha\sharp((C\sharp A)\sharp(\alpha\sharp B))) \neq C$	[531,735]
891	$C \neq C$	[535 (1) $\rightarrow$ 785 (1.2.2),735,735,551,566,337,337,668]
892	$\square$	[891.1,1.1]

SDS3

1	$x = x$	[axiom]

2	$(y \sharp y) \sharp (\alpha \sharp x) = x$	[axiom]
3	$(x \sharp \beta) \sharp (y \sharp y) = x$	[axiom]
4	$(x \sharp y) \sharp (z \sharp u) = (x \sharp z) \sharp (y \sharp u)$	[axiom]
5	$(A \sharp B) \sharp (((A \sharp C) \sharp \alpha) \sharp B) \neq C \ \mid \ \beta \neq \alpha$	[denial]
7,6	$\alpha \sharp (\alpha \sharp x) = x$	[2 (1) → 2 (1.1)]
8	$(x \sharp x) \sharp y = \alpha \sharp y$	[6 (1) → 2 (1.2)]
10	$(x \sharp \beta) \sharp \beta = x$	[3 (1) → 3 (1.2)]
13,12	$x \sharp (y \sharp y) = x \sharp \beta$	[10 (1) → 3 (1.1)]
16	$x \sharp (y \sharp z) = (\alpha \sharp y) \sharp ((\alpha \sharp x) \sharp z)$	[6 (1) → 4 (1.1)]
26	$(\alpha \sharp \beta) \sharp x = \alpha \sharp x$	[8 (1) → 8 (1.1),13]
29,28	$\alpha \sharp \beta = \beta$	[10 (1) → 8 (1)]
31,30	$(x \sharp y) \sharp (x \sharp z) = \alpha \sharp (y \sharp z)$	[4 (1) → 8 (1)]
33,32	$\beta \sharp x = \alpha \sharp x$	[26,29]
37,36	$x \sharp x = \beta$	[8 (1) → 10 (1.1),29,33,29]
41,40	$\beta = \alpha$	[28 (1) → 10 (1.1),37]
44,43	$x \sharp x = \alpha$	[36,41]
50,49	$(x \sharp \alpha) \sharp \alpha = x$	[10,41,41]
51	$(A \sharp B) \sharp (((A \sharp C) \sharp \alpha) \sharp B) \neq C$	[5,41,1]
71	$(\alpha \sharp ((A \sharp C) \sharp \alpha)) \sharp ((\alpha \sharp (A \sharp B)) \sharp B) \neq C$	[16 (1) → 51 (1)]
95	$((\alpha \sharp (A \sharp C)) \sharp \alpha) \sharp ((\alpha \sharp (A \sharp B)) \sharp B) \neq C$	[16 (1) → 71 (1.1),44,44]
97	$(C \sharp B) \sharp (\alpha \sharp B) \neq C$	[4 (1) → 95 (1),31,31,7]
99	$C \neq C$	[4 (1) → 97 (1),44,50]
100	$\square$	[99.1,1.1]

SDS4

1	$x = x$	[axiom]
2	$(x \sharp ((z \sharp z) \sharp z)) \sharp z = x$	[axiom]
3	$z \sharp ((z \sharp (z \sharp z)) \sharp x) = x$	[axiom]
4	$((x \sharp x) \sharp (x \sharp y)) \sharp ((y \sharp x) \sharp (x \sharp x)) = \alpha$	[axiom]
5	$(x \sharp y) \sharp (z \sharp u) = (x \sharp z) \sharp (y \sharp u)$	[axiom]
6	$\beta = \alpha$	[axiom]
7	$(A \sharp B) \sharp (((A \sharp C) \sharp \alpha) \sharp B) \neq C \ \mid \ \beta \neq \alpha$	[denial]
46	$(((x \sharp x) \sharp (x \sharp x)) \sharp ((x \sharp x) \sharp (x \sharp x))) \sharp \alpha = \alpha$	[4 (1) → 4 (1.2)]
49,48	$((x \sharp x) \sharp (y \sharp x)) \sharp ((x \sharp y) \sharp (x \sharp x)) = \alpha$	[5 (1) → 4 (1)]
51,50	$\alpha \sharp \alpha = \alpha$	[46,49]
65	$(\alpha \sharp (\alpha \sharp x)) \sharp ((x \sharp \alpha) \sharp \alpha) = \alpha$	[50 (1) → 4 (1.2.2),51]

68,67	$\alpha\sharp(\alpha\sharp x) = x$	[50 (1) $\rightarrow$ 3 (1.2.1.2),51]
70,69	$(x\sharp\alpha)\sharp\alpha = x$	[50 (1) $\rightarrow$ 2 (1.1.2.1),51]
73,72	$x\sharp x = \alpha$	[65,68,70]
104	$x\sharp((x\sharp\alpha)\sharp y) = y$	[3,73]
106	$(x\sharp(\alpha\sharp y))\sharp y = x$	[2,73]
110,109	$(x\sharp y)\sharp(x\sharp z) = \alpha\sharp(y\sharp z)$	[72 (1) $\rightarrow$ 5 (1.1)]
152,151	$(x\sharp y)\sharp(\alpha\sharp y) = x$	[104 (1) $\rightarrow$ 106 (1.1.2),73]
163	$x\sharp(\alpha\sharp((y\sharp\alpha)\sharp x)) = y$	[104 (1) $\rightarrow$ 151 (1.1)]
181	$x\sharp(\alpha\sharp(y\sharp x)) = y\sharp\alpha$	[106 (1) $\rightarrow$ 163 (1.2.2.1),73]
246,245	$(x\sharp y)\sharp\alpha = \alpha\sharp(y\sharp x)$	[151 (1) $\rightarrow$ 181 (1.2.2),110]
248,247	$\alpha\sharp((\alpha\sharp x)\sharp y) = x\sharp(\alpha\sharp y)$	[106 (1) $\rightarrow$ 181 (1.2.2),246]
276	$(A\sharp B)\sharp((\alpha\sharp(C\sharp A))\sharp B) \neq C$	[7,246,6]
288	$C \neq C$	[5 (1) $\rightarrow$ 276 (1),73,246,248,152]
289	$\square$	[288.1,1.1]

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